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NUMERICAL WAVE PROPAGATION SIMULATION

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ABSTRACT

Numerical modelling of seismic wave propagation, earthquake ground motion, seismic ambient noise and earthquake rupture dynamics are all crucial tasks in any comprehensive investigation of seismological and diverse geophysical processes in the Earth's interior as well. The increased resolution power of recent measurements requires faithfully numerically simulated seismic wavefields in realistic models of the Earth that take into account its complex rheology and geometry. This is well reflected by outstanding recent advances in numerical modelling. Three complementary approaches of paramount importance are the finite-difference, spectral-element and discontinuous Galerkin methods. In order to gain basic insights both on wave propagation features and on rupture dynamics, the boundary element methods are useful companions.

Keywords: seismic waves, numerical modelling, finite-difference, spectral-element, discontinuous-Galerkin, boundary elements.

INTRODUCTION

Wave-propagation configurations in seismology that we need to numerically simulate include a) material composition of a computational model, b) geometry of the model, and c) type of wave motion. Material constituents can be a single-phase solid (elastic, viscoelastic, elastoplastic, elastoviscoplastic, isotropic, anisotropic), single-phase fluid (liquid or gas, non-viscous or viscous) and multi-phase medium (poroelastic or poroviscoelastic with single- or two-phase pore fluid and zero or non-zero resistive friction). The model geometry can include geometry of borders of the computational region, free-surface topography, geometry of internal material interfaces and the faulting surfaces. Wavefield of interest can be acoustic or elastic produced by explosions, rupturing faults or random sources.

The three major factors a) – c) determine both A) structure of the controlling and constitutive equations in terms of temporal and spatial derivatives, and B) boundary conditions.

A) and B) are the factors that primarily determine time-space discretization of the model and a computational or numerical-modelling scheme.

Development of the numerical-modelling methods since the early 1970's reflects the above aspects of the problem. Earthquake seismology, structural seismology and exploration seismology have, however, in addition to many common aspects, also specific priorities. This is reflected by specific attention paid to, e.g., anisotropy, poroelasticity, small-scale heterogeneities, acoustic waves, seismic ambient noise, explosive sources, and rupturing faults.

The main general requirement is the optimal balance of three aspects – sufficiently faithful model, accuracy, and computational efficiency.

All these aspects, various needs and priorities powered the development of the numerical-modelling approaches. We address here three significant time-domain methods – finite-difference method (FDM), spectral-element method (SEM) and discontinuous Galerkin method (DGM). We briefly include the boundary integral equation method (BIEM). For the lack of space, we do not address pseudospectral, finite-element, finite-volume and other methods.

None of the methods is the best in terms of the general requirements and important wave-propagation configurations. A research team focused on investigating the Earth's structure or seismological phenomena should properly consider which method is the most suitable for the problem under scrutiny.

The books by Fichtner (2011), Moczo et al. (2014) and Igel (2017), dedicated to the numerical modelling of seismic wave propagation, can be recommended as related reading.

THE FINITE-DIFFERENCE METHOD

The FDM is, in fact, a global name for a large family of numerical-modelling methodologies based on the one unifying concept – approximating equations and boundary conditions by a FD scheme (FDS). The complete FD methodology must comprise a time-space grid, schemes for interior grid points, points at and near the Earth's free surface, points at and near the other borders of the grid, a rheological model of the medium, discrete representation of smooth and discontinuous material heterogeneity, discrete representation of a wavefield source. Individual FD methodologies that are being used differ in one or several these constituents. Rarely they comprise all the state-of-the-art ingredients. More often they prefer local (lab, group, single author) methodology due to tradition or the need to "sell" own solutions.

Development of FDM until 2013 is comprehensively presented in the book by Moczo et al. (2014). Most of terms and concepts used here are explained in the book. FDM has been intensively developing in recent years. We restrict here to several essential contributions. FDM has been developing considerably since then.

Quest for the Best Approximations and Stencils

The time-space staggered grid (SG) is fully consistent with the structure of velocity-stress (VS) equations for an isotropic medium. Because in many problems it is still reasonable to consider the isotropic medium, there is strong recent effort to find more accurate-and-efficient SG schemes for modelling acoustic and seismic waves. Interestingly, the most intensive and remarkable development has been ongoing in the exploration seismology (mainly Yang Liu, Mrinal Sen, their collaborators and several other Chinese seismologists). The key problem is to reach 1) the same high-order accuracy in space and time, in all propagation directions and in a wide range of wavenumbers, 2) optimal balance between increasing order and computational demands. Schemes with optional order of temporal and spatial accuracy and very good properties (accuracy and efficiency) were developed independently by Chen et al. (2021) for acoustic waves and by Zhou et al. (2021) for elastic waves. Both schemes are based on selective applications of the cross (axial) and cross-rhombus approximations to different first spatial derivatives, and standard 2^{nd}-order cross approximation to first temporal derivatives. Importantly, Zhou et al. (2021) used a decoupled VS formulation of the equation of motion and constitutive law. What remains to be properly addressed, is applicability of the new schemes to strongly heterogeneous media (as indicated, e.g., by Etemadsaeed et al. 2016) and geometrically complex boundary conditions.

Anisotropy and/or a deformation of the spatial grid imply that the equations can include spatial derivatives of all particle-velocity and stress components in all directions.

Lebedev (or fully-staggered) grid is consistent with the structure of the equations – spatial derivatives are as natural as on the SG and do not require interpolations. Lebedev grid in 3D is composed of 4 standard SG grids which decouple in an isotropic medium. de la Puente et al. (2014) used vertically deformed Lebedev grid and mimetic FD operators to include topography of the free surface.

The collocated grid is an alternative to Lebedev grid. Wei Zhang, Xiaofei Chen and their collaborators developed FD methodology to include free-surface (FS) topography, anisotropy and poroelasticity.

Challenging Free-surface Topography

Implementation of the FS topography is necessary because in many cases the FS topography can significantly affect seismic wavefield. At the same time, the implementation is still a nontrivial challenge.

There are three basic types of approaches: 1) fictitious (virtual) values of stress, particle velocity and material parameters above FS are used; the same stencil is used at all grid points (except, e.g., PML zone), 2) no fictitious values above FS; different stencils are used at grid points at and near FS, and in interior, 3) hybrid approach in which FDM is replaced by other method at and near FS.

The first approach is probably best represented by Zhang et al. (2012a,b) and Sun et al. (2016, 2018, 2019) who developed the traction-image method using the FS-conforming collocated curvilinear grid, dispersion-relation-preserving MacCormack scheme, and conservative form of divergence. The very important and distinctive feature of the approach is the fact that the curved grid is deformed in all three directions in order to conform with geometry of FS, that is, not only in the vertical direction. This is one of the reasons, together with imaging traction (nor just stress) components, why Zhang et al. can reach stability and accuracy even for rough topography. Building up on this approach, Zang et al. (2021) developed a promising overset grid FD method which combines background rectangular grid with a few layers of a generally deformed grid fitting topography of FS. Gao et al. (2015) presented a staggered-grid immersed-boundary approach in the 3D follow-up of the approach by Lombard et al. (2008). They calculate fictitious values in vacuum using boundary and compatibility conditions and inject them into an interior scheme applied at and near FS.

In the approach with no fictitious values, de la Puente et al. (2014) presented a FD scheme based on the velocity-stress formulation but overcoming limitations of the standard staggered grid. They use a vertical deformation to conform FS topography, Lebedev grid, and mimetic FD operators. Shragge & Konuk (2020) used tensorial formulation to develop a novel system of semianalytic velocity-stress equations for a family of vertically deformed grids and a corresponding FS condition. They use Lebedev grid and mimetic approximations to develop the corresponding FD scheme.

Lisitsa et al. (2016) developed a hybrid approach combining DGM on a polyhedral mesh and FDM on rectangular non-staggered and standard staggered grids in order to implement both complex near-surface material heterogeneity and FS topography.

Material Heterogeneity and Rheology

Implementation of sharp internal material interfaces is another challenging task in FD modelling. It is important to have sufficiently accurate implementation with sub-cell resolution. The latter means that a FD scheme should be capable to “sense” an arbitrary shape and position of the interface in the uniform grid. Including different positions within one grid cell. Two important approaches have been developed recently.

Mittet (2017) correctly noted that the wavefields and material-parameter fields should be treated consistently. He suggested to use the wavenumber representation of a properly band-limited Heaviside step function (to the maximum wavenumber allowed for the grid) and transform it to the space domain. Alternatively, a fine-grid model can be created that is low-pass filtered to remove wavenumbers inappropriate for the coarse simulation grid. The fine grid is then sampled at the required coordinates for the coarse grid. Mittet (2021) elaborated clever analysis of accuracy of the implementation.

Moczo et al. (2014, 2019), Kristek et al. (2017, 2019) and Gregor et al. (2021) developed orthorhombic representations of strongly heterogeneous elastic, viscoelastic a poroelastic media with material interfaces. The main idea is to best represent smooth and discontinuous material heterogeneity in the FD modelling using the standard staggered-grid scheme (4th-order accurate in space, 2nd-order accurate in time). All schemes keep computational efficiency of the corresponding velocity-stress or velocity-pressure-stress staggered grid schemes for smoothly and weakly heterogeneous media. The viscoelastic medium has rheology of the generalized Maxwell body (GMB-EK, equivalent to the generalized Zener body). The poroelastic medium can have zero resistive friction or non-zero resistive friction or JKD frequency-dependent permeability and resistive friction. All FD schemes are capable of sub-cell resolution. Accuracy, including sub-cell resolution capability, was tested against SEM, DGM, analytical and semi-analytical methods.

Kristek et al. (2018) and Moczo et al. (2019) demonstrated the need and importance of geometrically and rheologically complex models in the quest for identification of key structural parameters and key characteristics of earthquake ground motion. These are necessary in an effective identification of the potential of local surface sedimentary structures to cause anomalous ground motions and site effects of earthquakes.

Alkhimenkov et al. (2021) developed a comprehensive von Neumann stability analysis for a class of FD schemes for Biot's equations for a poroelastic medium.

Models consisting of homogeneous or weakly heterogeneous layers/blocks often fail to reproduce complexity/features of observed seismic wave fields, particularly at higher frequencies. This is partly because they do not capture small-scale variations. The small-scale heterogeneities can be modelled as correlated random fields. Small-scale heterogeneities have been considered in several recent studies. For example, Bydlon & Dunham (2015) studied the effects of geometric and material heterogeneity on the rupture process and resulting high-frequency ground motions in the near-fault region, Vyas et al. (2018) investigated coherence of Mach waves radiated by supershear ruptures, Savran & Olsen (2019) simulated ground motions during 2008 Chino Hills earthquake, Hirakawa et al. (2016) and Scalise et al. (2020) studied generation of shear waves during underground explosions.

Small-scale heterogeneities pose a new modelling challenge because they can formally require smaller grid spacing and/or time step. Yet, the lowest S-wave speed occurs only at a few points. Grid spacing is formally further constrained by requirement of proper capturing effects of small-scale heterogeneities on wavefield. Both above aspects of modelling are still to be investigated.

Wavefield Excitation

Spontaneous rupture propagation poses a specific challenge for FDM. The well-known traction-at-split node (TSN) method is still considered the most accurate approach. Zhang et al. (2014) implemented TSN into 3D collocated-grid FD scheme for non-planar ruptures. Zhang et al. (2016) improved implementation of the boundary conditions at intersections of fault with FS, and Zhang et al. (2020) optimized the code for GPUs.

Koene et al. (2020) developed a consistent implementation of point sources at arbitrary position in the spatial FD grid.

Discontinuous Grids

Discontinuous grid can significantly reduce the number of arithmetic operations and memory. As explained by Moczo et al. (2014), the effort had been for a long time focused on interpolating field values at missing grid positions. Recently it is mainly about the stability of a contact of the finer and coarser grids in heterogeneous medium. A fully satisfactory solution is still to be found. Another important aspect is the local time-space refinement. We refer to Kostin et al. (2015), Li et al. (2015), Fan et al. (2015), Nie et al. (2017) for more details.

THE SPECTRAL-ELEMENT METHOD

Wave Propagation

Fundamentals of SEM applied to the numerical simulation of seismic wave propagation for Earthquake Ground Motion (EGM) prediction are summarized in the book of Moczo et al. (2014) and extensive reviews can be found in the references therein.

As often stated, SEM combines the geometrical flexibility of FEM with the accuracy of spectral methods. This very general assertion has been refined and somewhat softened in recent studies. For example, Chaljub et al. (2015) have evidenced that despite the fact that SEM has the ability to account for physical discontinuities in material properties, just like FEM, the lack of flexibility related to the use of tensorized elements (deformed quadrangles in 2D, hexahedra in 3D) can lead to large numerical errors whenever interfaces cannot be explicitly discretized in the meshing process. This is of particular importance for the accurate simulation of surface waves diffracted off strong lateral contrasts close to the surface (e.g. basin or valley edges). More generally, the mis-representation of small-scale features of the propagation medium at the discrete level is one of the known source of epistemic uncertainty for EGM assessment with SEM, as it is with any other grid-based method. A way to solve the sub-cell representation issue is to design effective media through homogenization, generally in a pre-processing stage. The most general approach, so-called Two-Scale Homogenization (TSH), leads to fully anisotropic effective media which can easily be implemented in SEM (see Capdeville et al. 2020 for a review). TSH can in principle be applied to most of the realistic configurations considered for EGM applications (see Cupillard & Capdeville, 2018), although it still has to be evaluated in situations which combine small-scale heterogeneities close to the free-surface (possibly including variations of the surface itself) superimposed on a background medium with strong velocity contrasts.

Whenever the propagation medium can be successfully homogenized, Chao et al. (2020) show that very high polynomial orders can be used in SEM (up to 40!) without altering its computational efficiency. This suggests that the process of homogenization yields smooth wavefields, the discretization of which can then benefit from the spectral accuracy of SEM.

In other situations, mainly when the geometry can neither be accurately represented by hexahedra nor simplified through homogenization, SEM can be efficiently coupled to other methods, in particular FEM and DGM, which also rely on variational formulations. Such example of hybrid methods, using a mortar element coupling approach, has been proposed by Brun et al. (2021) to model the seismic response of concrete dams in the linear and non-linear regimes. Another motivation to couple SEM with DGM arises for seismic wave propagation in media with fluid regions having either complex geometries or non-linear dynamics (Terrana et al., 2018; Brissaud et al., 2017). One reason for such coupling is that the discretization of fluid regions in SEM requires a specific, potential formulation of the wave equation and an explicit representation of the fluid-solid interfaces which cannot be handled through homogenization.

Earthquake Rupture Dynamics

Building up on the work of Komatitsch and Vilotte (1998) on SEM for wave propagation, the capability to simulate dynamic earthquake ruptures in SEM was developed in 2D by Ampuero (2002) and implemented in the open-source code SEM2DPACK (https://github.com/jpampuero/sem2dpack), and by Festa and Vilotte (2006). It was extended to 3D by Kaneko et al. (2008) and implemented in the open-source code SPECFEM3D (https://github.com/geodynamics/specfem3d).

A non-trivial achievement of these early works was to show that SEM is well adapted for highly non-linear problems with non-smooth solutions (discontinuous high-order derivatives near the rupture front). The method was later extended to earthquake problems with non-linear rheologies in the medium surrounding the fault (Gabriel et al., 2013). Developments in the past ~5 years are summarized below, related to methodological aspects and to the insights on earthquake physics they have enabled.

Recent methodological developments of SEM for rupture dynamics include: the extension to non-planar fault systems discretized by unstructured meshes (Galvez et al., 2014), the simulation of dynamic rupture interacting with off-fault continuum damage (Xu et al., 2015), the modeling of transient gravity signals produced by earthquakes (Harms et al., 2015), an adjoint method for finite source inversion in heterogeneous media (Somala et al., 2018), and the coupling between dynamic SEM and quasi-static BEM (Boundary Element Method) for the simulation of earthquake cycles (Galvez et al., 2020).

Developments in progress include the joint simulation of earthquake cycles, fault growth and evolution of fault damage zones, which is a highly non-linear problem spanning a very wide range of time-scales and length-scales. An outstanding bottleneck in applications of SEM to earthquake dynamics can be mesh generation. The method is based on the so-called split-node approach (e.g. Day et al., 2005) and hexahedral spectral elements. It requires specialized treatment during the generation of the spectral element mesh, which can be complicated in geometries involving narrow dip angles, such as branched fault systems and shallow-dipping faults. Narrow angles also limit the time step size, increasing the computational cost, which could be addressed by local time stepping, i.e. integration schemes that allow for different time step sizes in each element (Rietmann et al., 2017).

SEM has enabled several recent advances in earthquake physics, including: a new theory describing the dynamics of very large earthquakes (Weng and Ampuero, 2019), the effect of mixed-mode rupture (oblique slip) on earthquake speeds (Weng and Ampuero, 2020), the effects of fault damaged zones on dynamic rupture (Oral et al., 2019; Huang et al., 2014, 2015), the limiting effect of seismogenic width on the thickness of fault damage zones (Ampuero and Mao, 2017) and on the distance ruptures can jump across fault stepovers (Bai and Ampuero, 2017). SEM has also facilitated synergistic studies of the 2011 Tohoku, Japan earthquake, by combining geophysical observations, laboratory experiments and numerical simulations (Galvez et al., 2014, 2016; Hirono et al., 2016; Tsuda et al., 2016).

THE DISCONTINUOUS GALERKIN METHOD

The Main Methodological Advances

The Discontinuous Galerkin (DG) method was first introduced for the neutron transport equation using high-order Runge-Kutta integration schemes and subsequently extended to general hyperbolic systems (Hesthaven and Warburton, 2008).

Due to the spatially local character of its discrete high-order accurate operators, DG allows to use boundary conforming curvilinear (Warburton, 2013) or unstructured meshes composed of triangles and tetrahedra which simplifies accounting for complex geological structures or complex topography, such as volcanoes, sedimentary basins, fault and fracture zones, geological interfaces and sharp impedance contrasts (e.g., Mercerat & Glinsky, 2015; Gabriel et al., 2020). DG's use of numerical fluxes, which do not impose any field continuity across their boundaries, allows to naturally include non-linear interface conditions, arising, e.g., in dynamic earthquake rupture problems (Tago et al., 2012; Pelties et al., 2012). For modelling wave phenomena, DG schemes are advantageous as they exhibit low numerical dispersion and the flux-based formulation additionally introduces numerical dissipation which scales in accordance with the cell size and polynomial degree (Kopriva et al., 2017).

The cell-wise local character of DG can be readily combined with an Arbitrary high-order DERivative (ADER) time integrator, leading to high-order accuracy in time within a single-step. This approach was recently extended for non-linear problems with a-posteriori subcell limiting and finite volume (FV) schemes (Reinarz et al., 2019).

Recently, DG schemes have become increasingly popular for seismic wave propagation and dynamic earthquake rupture simulations on large-scale high-performance computing (HPC) infrastructure (Wilcox et al., 2010; Breuer et al., 2014; Heinecke et al., 2014). This increase in interest using DG in HPC applications is attributed to (i) the easy and efficient implementation of local time stepping schemes (Uphoff et al., 2017); (ii) straight-forward inclusion of adaptive mesh refinement (Burstedde et al., 2011); and (iii) on-node hardware optimisations, for example through exploiting equivalent sparsity patterns and optimal index permutations for temporary tensors (Uphoff & Bader, 2020). The optimisation of the ADER-DG scheme has been extended to different rheologies including viscoelasticity and anisotropy (Uphoff & Bader, 2016, Wolf et al., 2020).

Using DG, large-scale and geometrically and rheologically complicated wave propagation and dynamic rupture simulations can be performed in a few hours on today's (and tomorrow's, e.g., Dorozhinskii & Bader, 2021) supercomputers.

Earthquake Rupture Dynamics

Understanding the dynamics of earthquakes and faults poses a multi-scale, multi-physics problem with profound societal implications, including secondary effects with unforeseen hazard complexity. Earthquakes can be represented as frictional shear fracture of brittle solids under compression along internal boundaries of pre-existing weak fault interfaces. Rupture dynamics computations can constrain earthquake scenarios based on constitutive laws, initial stress conditions, lithology, and fault geometries. This provides a physics-based understanding of how earthquakes start, propagate, and stop. During the highly non-linear interaction of frictional failure and seismic wave propagation (e.g., Gabriel et al., 2012) computational models may, in addition, include multi-physics processes on- and off the fault, like off-fault deformation or thermal pressurization of rock pore fluids (e.g., Wollherr et al., 2018).

DG methods have been particularly impactful in this context having the potential to overcome the limitations of short-term (incomplete) earthquake data by incorporating heterogeneous field data and laboratory measured rock behaviour across geometrically complex faults. Community efforts, inspired by the wave propagation community, have been rigorously verifying various numerical techniques for dynamic rupture problems of increasing complexity (Harris et al., 2018), but also highlighted the difficulties of treating complexities such as branching and intersecting fault systems. In contrast to the typically applied traction at split-node approach, DG methods solve for the frictional sliding via fault-flux approximations. In the ADER-DG method the inverse Riemann problem (LeVeque, 2002) is solved at fault element interfaces, in which the exact solution is modified to incorporate frictional boundary conditions such as slip-weakening or rate-and-state-dependent behaviour. Solving the inverse Riemann problem inherits the favourable numerical properties from the exact Riemann solver or Godunov upwind flux which exhibits a very selective numerical dissipation for first-order hyperbolic problems (Chan & Warburton, 2017) subduing spurious high frequency oscillations while minimally affecting physically meaningful frequencies (Pelties et al., 2014).

High-resolution strong ground motion and physics-based dynamic earthquake rupture modelling has been advanced significantly by, and in turn contributed to, large-scale HPC: Recent computational advances now allow to capture non-linear rupture dynamics on complex faults up to the scale of megathrust events. DG earthquake simulation “hero runs” reach close to 50% peak performance on current multi-PFLOPS supercomputing systems (Uphoff et al., 2017) and cover physical scales which can be linked to the mechanical processes of tsunami generation and geodynamic processes (van Zelst et al., 2019; Ulrich et al., 2020; Madden et al., 2020). Such large-scale dynamic rupture models can simultaneously capture seismic, geodetic (and tsunami) observations during moderate (Palgunadi et al., 2020) and large earthquakes (Wollherr et al., 2019; Ulrich et al., 2019a,b, 2020).

Future Developments

HPC empowered computational seismology and earthquake modelling is a key tool to better understand the structure of Earth’s interior and sources of seismic energy. It is becoming an important part of the rapid earthquake response toolset by delivering physics-driven interpretations that can be integrated synergistically with data-driven efforts. On-going challenges include computational efficiency (resolving smaller scale heterogeneities affecting the high frequencies of the wave field), including the multi-physics processes of fault zone complexity, addressing model sensitivities and the need for open-source community solutions. Earthquake fault zones are more complex, both geometrically and rheologically, than an idealised infinitely thin plane embedded in linear elastic material, potentially requiring to fully model volumetric fault zone shearing during earthquake rupture, which includes spontaneous partition of fault slip into intensely localized shear deformation within weaker (possibly cohesionless/ultracataclastic) fault-core gouge and more distributed damage within fault rocks and foliated gouges (Gabriel et al., 2020). Combination with emerging machine learning approaches may close the gap of data quality in real time conditions, boosting our understanding of earthquake source processes and the associated ground shaking. In the near future, HPC allows exploring multitudes of scenarios allowing for example for the use of dynamic rupture simulations for Bayesian inversions (e.g., Gallovic et al., 2019). To understand error propagation and to develop probability densities for the parameters related to fundamental earthquake physics as well as physics-based monitoring and megathrust hazard assessment of the future, incorporating uncertainties is the critical next step.

THE BOUNDARY INTEGRAL APPROACHES

Integral Equation Methods (BEM & IBEM)

Within the linear elasticity, the Somigliana’s representation theorem relates the displacement field with a boundary integral of both displacements and tractions combined with the Green’s function and the associated tractions.

Two main formulations have been proposed. The so-called direct boundary element method (DBEM) stems directly from Somigliana's theorem and is based on reciprocity. Much of the terminology is related with the finite-element method (FEM). The other approach is the indirect boundary element method (IBEM). It is a consequence of the linearity of problem as the field is simply composed by the superposition of distributed loads which are intermediate unknowns. It can be seen as the mathematical transcription of Huygens' principle.

Although the description of the boundary methods can be made both in time and frequency, the latter approach has been prevailed in seismology. The discretization of the boundaries, the application of boundary conditions and the use of Fourier transform leads to DBEM (Kawase, 1988) or IBEM (Sánchez-Sesma and Campillo, 1991; Sánchez-Sesma and Luzón, 1995) numerical approaches.

A complete review on the use of BEM for seismic problems can be found in Bouchon & Sánchez-Sesma (2007). Perton et al. (2016) presented an implementation of the IBEM that allows to simulate elastic wave propagation in complex configurations made of embedded regions that are homogeneous with irregular boundaries or with flat layers.

Despite the elegance of these boundary formulations, they are limited because the Green's function is only available for homogeneous or for constant gradient media. Thus, the modelling of the realistic settings and rheologies is beyond its capacities. However, the versatility and the intrinsic economy of boundary formulations make them attractive to test and validate the volume approaches discussed herein.

CONCLUSIONS

The recent methodological advances in numerical modelling of seismic wave propagation are remarkable. As desired and expected, they make it possible to study the Earth's structure and processes with the unprecedented accuracy.

On the other hand, they, together with ongoing efforts, clearly indicate that there is a vast space for necessary improvements. Sufficiently realistic complexity of models of the Earth's interior and seismological processes, optional level of accuracy and feasible computational efficiency for desirable/necessary space, time and frequency domains make together still ambitious and challenging task for developers of the methods and a dream for those who need numerical modelling to investigate Earth's structure and processes.

A dream numerical modelling code should allow for a geometrically complex model potentially made of blocks/layers of different rheologies as well as for a realistic wavefield generation. These aspects of the wavefield-model configuration should not be compromised by insufficient accuracy and insufficient computational efficiency. Such codes may be specific for small-scale and regional Earth's model.

It is very obvious that such numerical-modelling codes are still to be developed. There are many unsolved partial aspects in each of the recent numerical methods. It is important to continue developing each of the recent significant methods. This is because it is very likely that a truly universal (that is, suitable for all seismological problems) dream method will not be developed – at least not in the close future.

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