On the rheological models used for time-domain methods of seismic wave propagation

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[1] After publications by *Emmerich and Korn* [1987] and *Carcione et al.* [1988a, 1988b] authors who implemented realistic attenuation in the time-domain methods decided for either of two rheological models – generalized Maxwell body (as defined by Emmerich and Korn) or generalized Zener body. Two parallel sets of papers and mathematical formalisms developed during the years. We have not found any comments on the other rheology. Therefore, we review both models and show that, in fact, they are equivalent. We also derive material-independent anelastic functions. **Citation:** Moczo, P., and J. Kristek (2005), On the rheological models used for time-domain methods of seismic wave propagation, *Geophys. Res. Lett.*, *32*, L01306, doi:10.1029/2004GL021598.

1. Introduction

[2] The rheological behavior of the Earth's materials can be modeled using viscoelastic models because it combines behaviors of elastic solids and viscous fluids. The observations show that the internal friction in the Earth is nearly constant over the seismic frequency range. This has been well recognized since the work of *Liu et al.* [1976].

[3] For a linear isotropic viscoelastic material, the stress-strain relation is given by Boltzmann principle. In a simple scalar notation,

$$\sigma(t) = \int_{-\infty}^{t} \psi(t-\tau)\dot{\varepsilon}(\tau)d\tau \tag{1}$$

where $\sigma(t)$ is stress, $\dot{\varepsilon}(t)$ time derivative of strain, and $\psi(t)$ stress relaxation function defined as a stress response to Heaviside unit step function in strain. The stress at a given time t is determined by the entire history of the strain until time t. Mathematically, the integral in equation (1) represents a time convolution of the relaxation function and strain rate. Using symbol * for the convolution and applying the convolution's property, we have

$$\sigma(t) = \psi(t) * \dot{\varepsilon}(t) = \dot{\psi}(t) * \varepsilon(t). \tag{2}$$

Since $\psi(t)$ is the stress response to a unit step function in strain, its time derivative $M(t)$ is the stress response to the Dirac δ -function in strain. Therefore

$$\sigma(t) = M(t) * \varepsilon(t) \tag{3a}$$

$$M(t) = \dot{\psi}(t) \tag{3b}$$

Let $\mathcal{F}$ be the direct and $\mathcal{F}^{-1}$ the inverse Fourier transforms:

$$\mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\exp(-i\omega t)\,dt,$$

$$\mathcal{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\exp(i\omega t)\,d\omega$$

with ω being the angular frequency. The Fourier transform of equation (3a) gives

$$\sigma(\omega) = M(\omega)\cdot\varepsilon(\omega) \tag{4}$$

where $M(\omega)$ is a complex, frequency-dependent viscoelastic modulus. From equations (3a), (3b), and (4) we get

$$\psi(t) = \mathcal{F}^{-1}\left\{\frac{M(\omega)}{i\omega}\right\}. \tag{5}$$

An instantaneous elastic response of the viscoelastic material is given by the unrelaxed (elastic) modulus M_U, a long-term equilibrium response is given by the relaxed modulus M_R; $M_U = \lim_{t\to 0} M(t)$, $M_R = \lim_{t\to\infty} M(t)$. In the frequency domain, $M_U = \lim_{\omega\to\infty} M(\omega)$, $M_R = \lim_{\omega\to 0} M(\omega)$. The modulus defect or relaxation of modulus is

$$\delta M = M_U - M_R. \tag{6}$$

The quality factor $Q(\omega)$ is defined by $Q(\omega)$ = Re $M(\omega)$/Im $M(\omega)$. Detailed introductions to the theory of viscoelasticity can be found e.g., in work by *Biot* [1954], *Fung* [1965], *Ben-Menahem and Singh* [1981], and *Carcione* [2001].

[4] Equation (4) indicates that the incorporation of the linear viscoelasticity and consequently attenuation into the frequency-domain computations is easy: real frequency-independent moduli are replaced by complex, frequency-dependent quantities (the correspondence principle). At the same time, a numerical integration of the time-domain stress-strain relation (1) is practically intractable due to large computer time and memory requirements. Therefore modelers incorporated oversimplified $Q(\omega)$ laws.

[5] The breakthrough in incorporating realistic attenuation into the time-domain computations was made by *Day and Minster* [1984]. They pointed out that if $M(\omega)$ is a rational function with the nth-order denominator, the inverse Fourier transform of equation (4) gives an nth-order differential equation for $\sigma(t)$, which can be eventually numerically solved much more easily than the convolution integral. They, however, assumed that, in general, the viscoelastic modulus is not a rational function. Therefore they suggested approximating a viscoelastic modulus by a nth-order rational function and determining its coefficients

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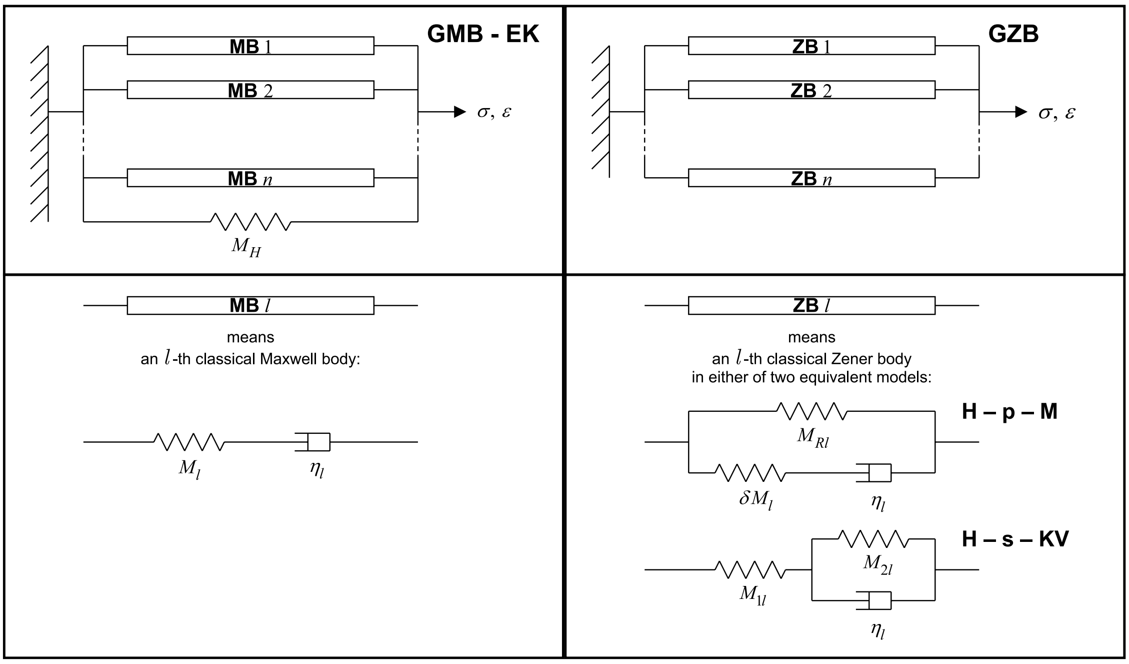

Figure 1. (left) Generalized Maxwell Body, GMB-EK, defined by *Emmerich and Korn* [1987]. M_H and M_l denote elastic moduli, η_l viscosity. (right) Generalized Zener Body, GZB. For a classical Zener body (standard linear body) there are two equivalent models: H-p-M - Hooke element connected in parallel with Maxwell body, and H-s-KV - Hooke element connected in series with Kelvin-Voigt body. In the H-p-M model it is easier to recognize the relaxed modulus M_{Rl} and modulus defect δM_l. M_{1l} and M_{2l} in the H-s-KV model denote elastic moduli, η_l in both models viscosity.

by the Padé approximant method. They obtained n ordinary differential equations for n additional variables, which replace the convolution integral. The sum of the internal variables multiplied by the unrelaxed modulus gives an additional viscoelastic term to the elastic stress. *Day and Minster* [1984], in fact, also indirectly suggested the future evolution – a direct use of the rheological models whose $M(\omega)$ is a rational function.

[6] In response to work of *Day and Minster* [1984], *Emmerich and Korn* [1987] realized that an acceptable relaxation function corresponds to rheology of what they defined as the generalized Maxwell body – n Maxwell bodies and one Hooke element (elastic spring) connected in parallel - see Figure 1. Because the generalized Maxwell body in the literature on rheology is defined without the additional single Hooke element, we denote the model of *Emmerich and Korn* [1987] as GMB-EK. The viscoelastic modulus of the GMB-EK is a rational function. *Emmerich and Korn* [1987] obtained for the new variables similar differential equations as *Day and Minster* [1984]. *Emmerich and Korn* [1987] demonstrated that their approach is better than the approach based on the Padé approximant method both in accuracy and computational efficiency. Independently, *Carcione et al.* [1988a, 1988b], in accordance with the approach of *Liu et al.* [1976], assumed the generalized Zener body (GZB) - n Zener bodies, that is, n standard linear bodies, connected in parallel; see Figure 1. *Carcione et al.* [1988a, 1988b] developed a theory for the GZB and introduced term memory variables.

[7] After publications by *Emmerich and Korn* [1987] and *Carcione et al.* [1988a, 1988b] many different authors decided either for the GMB-EK [e.g., *Emmerich*, 1992; *Fäh*, 1992; *Moczo and Bard*, 1993; *Moczo et al.*, 1997; *Kay and Krebes*, 1999] or GZB [e.g., *Robertsson et al.*, 1994; *Blanch et al.*, 1995; *Xu and McMechan*, 1995; *Robertsson*, 1996; *Hestholm*, 2002]. In both cases the authors followed the corresponding mathematical formalisms. In the aforementioned and many other papers we have found neither direct comparison nor any comment on the other rheology and the corresponding algorithms. Therefore, we briefly review both rheological models and show their relation.

2. The GZB and GMB-EK Rheological Models

[8] In Table 1 we summarize frequency-domain rules for linear rheological models. Using the rules it is easy to find the stress-strain relation in the frequency domain and thus the viscoelastic modulus.

[9] For the GMB-EK we find

$$M(\omega) = M_H + \sum_{l=1}^{n} \frac{iM_l\omega}{\omega_l + i\omega} \tag{7}$$

with relaxation frequencies

$$\omega_l = M_l/\eta_l \ ; \quad l = 1, \ldots, n.$$

The relaxed and unrelaxed moduli are $M_R \equiv \lim_{\omega\to 0} M(\omega) = M_H$ and $M_U \equiv \lim_{\omega\to\infty} M(\omega) = M_R + \sum_{l=1}^{n} M_l$. Since, equation (6), $M_U = M_R + \delta M$, we can see that

$$M_l = \delta M_l.$$

Table 1. Frequency-Domain Rules for Linear Rheological Models

Element	Stress-Strain Relation	
Hooke (spring)	$\sigma(\omega) = M \cdot \varepsilon(\omega)$, M - elastic modulus	
Stokes (dashpot)	$\sigma(\omega) = i\omega\eta \cdot \varepsilon(\omega)$, η - viscosity	
Connection	σ	ε
in series	equal	additive
in parallel	additive	equal

Without loss of generality we can consider $\delta M_l = a_l \delta M$; $\sum_{l=1}^{n} a_l = 1$. Then

$$M(\omega) = M_R + \delta M \sum_{l=1}^{n} \frac{i a_l \omega}{\omega_l + i\omega}. \tag{8}$$

Using equation (5) we obtain the relaxation function

$$\psi(t) = \left[M_R + \delta M \sum_{l=1}^{n} a_l e^{-\omega_l t} \right] \cdot H(t) \tag{9}$$

where $H(t)$ is the Heaviside unit step function. The above formulas were presented by *Emmerich and Korn* [1987].

[10] From the two equivalent models of the GZB (Figure 1, right) we choose the one in which a single ZB is of the H-p-M type (Hooke element in parallel with Maxwell body). This is because we can immediately see the meaning (M_{Rl}, δM_l) of the elastic moduli of both Hooke elements in each ZB. Using the rules in Table 1 we get

$$M(\omega) = \sum_{l=1}^{n} M_{Rl} \frac{1 + i\tau_{\varepsilon l}\omega}{1 + i\tau_{\sigma l}\omega}$$

with relaxation times

$$\tau_{\varepsilon l} = \frac{\eta_l}{\delta M_l} \frac{M_{Ul}}{M_{Rl}} \quad , \quad \tau_{\sigma l} = \frac{\eta_l}{\delta M_l} \quad , \quad \frac{\tau_{\varepsilon l}}{\tau_{\sigma l}} = \frac{M_{Ul}}{M_{Rl}}$$

and

$$M_{Ul} = M_{Rl} + \delta M_l.$$

The unrelaxed and relaxed moduli are $M_R \equiv \lim_{\omega \to 0} M(\omega) = \sum_{l=1}^{n} M_{Rl}$ and $M_U \equiv \lim_{\omega \to \infty} M(\omega) = \sum_{l=1}^{n} M_{Rl} \frac{\tau_{\varepsilon l}}{\tau_{\sigma l}} = M_R + \sum_{l=1}^{n} \delta M_l$. The relaxation function is

$$\psi(t) = \left\{ \sum_{l=1}^{n} M_{Rl} \left[1 - \left(1 - \frac{\tau_{\varepsilon l}}{\tau_{\sigma l}} \right) e^{-t/\tau_{\sigma l}} \right] \right\} \cdot H(t) \tag{10}$$

Defining [*Carcione*, 2001]

$$M_{Rl} = \frac{1}{n} M_R \tag{11}$$

we get

$$M(\omega) = \frac{M_R}{n} \sum_{l=1}^{n} \frac{1 + i\tau_{\varepsilon l}\omega}{1 + i\tau_{\sigma l}\omega},$$
$$\psi(t) = M_R \left[1 - \frac{1}{n} \sum_{l=1}^{n} \left(1 - \frac{\tau_{\varepsilon l}}{\tau_{\sigma l}} \right) e^{-t/\tau_{\sigma l}} \right] \cdot H(t). \tag{12}$$

Formulas (11) and (12) were presented by *Carcione* [2001]. Note that *Liu et al.* [1976], in generalizing the strain-stress relation for one ZB (equation 16 in their paper) to the relation for the GZB (equation 22 in their paper), introduced an error, which then has been repeated in the following papers dealing with the incorporation of the attenuation based on the GZB - even after *Carcione* [2001] published correct formulas for the relaxation function and modulus. In all papers we found, there is the same error – the missing factor $1/n$ in the viscoelastic modulus and relaxation function.

3. The Relation Between the GZB and GMB-EK

[11] Consider again the ZB (H-p-M) model. Using the rules (Table 1) for the ZB l gives

$$\sigma_l(\omega) \cdot \left(\frac{1}{\delta M_l} + \frac{1}{i\eta_l\omega} \right) = \left(1 + \frac{M_{Rl}}{\delta M_l} + \frac{M_{Rl}}{i\eta_l\omega} \right) \cdot \varepsilon(\omega). \tag{13}$$

Defining relaxation frequency

$$\omega_l = \delta M_l / \eta_l$$

and rearranging equation (13) we get

$$\sigma_l(\omega) = M_l(\omega) \cdot \varepsilon(\omega) \quad ; \quad M_l(\omega) = M_{Rl} + \frac{i\delta M_l \omega}{\omega_l + i\omega}.$$

For n ZB (H-p-M) connected in parallel, that is, for the GZB (Figure 1, right) the stress is

$$\sigma(\omega) = \sum_{l=1}^{n} \sigma_l(\omega) = \left[\sum_{l=1}^{n} M_l(\omega) \right] \cdot \varepsilon(\omega)$$

and thus

$$M(\omega) = \sum_{l=1}^{n} M_{Rl} + \sum_{l=1}^{n} \frac{i\delta M_l \omega}{\omega_l + i\omega}.$$

We easily find that

$$M_R = \sum_{l=1}^{n} M_{Rl} \quad , \quad M_U = M_R + \sum_{l=1}^{n} \delta M_l.$$

Since $M_U = M_R + \delta M$, without loss of generality we can consider

$$\delta M_l = a_l \delta M \quad ; \quad \sum_{l=1}^{n} a_l = 1$$

and get

$$M(\omega) = M_R + \delta M \sum_{l=1}^{n} \frac{i a_l \omega}{\omega_l + i\omega}. \tag{14}$$

We see that for the GZB (H-p-M), Figure 1 (right), we obtained exactly the same $M(\omega)$ as it has been obtained by *Emmerich and Korn* [1987] for their GMB-EK (Figure 1, left), equation (8) in this paper. It is also easy to get the same viscoelastic modulus for the GZB (H-s-KV). We can also rewrite non-simplified $\psi(t)$ for the GZB, equation (10), into the form of $\psi(t)$ for the GMB-EK, equation (9), without any simplification. In other words, the rheology of the GMB-EK and GZB is one and the same.

4. Derivation of the Material-Independent Anelastic Functions

[12] Having recognized the equivalence of the GMB-EK and GZB models, we can finish with derivation of the anelastic functions (memory variables). *Day and Minster* [1984], *Emmerich and Korn* [1987], and *Carcione et al.* [1988a, 1988b] introduced their memory variables as material-dependent. *Kristek and Moczo* [2003] introduced material-independent anelastic functions. They explained why and showed numerical tests. They did not show the derivation directly from the basic rheological equations for the GMB-EK.

[13] Rewrite the viscoelastic modulus (14) and relaxation function (9) using the unrelaxed modulus,

$$M(\omega) = M_U - \delta M \sum_{l=1}^{n} \frac{a_l \omega_l}{\omega_l + i\omega}$$

and

$$\psi(t) = \left[M_U - \delta M \sum_{l=1}^{n} a_l \left(1 - e^{-\omega_l t}\right) \right] \cdot H(t),$$

and obtain the time derivative of the relaxation function

$$\begin{aligned} M(t) = \dot{\psi}(t) = & -\delta M \sum_{l=1}^{n} a_l \omega_l e^{-\omega_l t} \cdot H(t) \\ & + \left[M_U - \delta M \sum_{l=1}^{n} a_l \left(1 - e^{-\omega_l t}\right) \right] \cdot \delta(t). \end{aligned} \tag{15}$$

Inserting equation (15) into equation (3a) gives

$$\begin{aligned} \sigma(t) = & -\int_{-\infty}^{t} \delta M \sum_{l=1}^{n} a_l \omega_l e^{-\omega_l (t-\tau)} \cdot H(t-\tau) \cdot \varepsilon(\tau) d\tau \\ & + \int_{-\infty}^{t} M_U \cdot \delta(t-\tau) \cdot \varepsilon(\tau) d\tau \\ & - \int_{-\infty}^{t} \delta M \sum_{l=1}^{n} a_l \left(1 - e^{-\omega_l (t-\tau)}\right) \cdot \delta(t-\tau) \cdot \varepsilon(\tau) d\tau \end{aligned}$$

and

$$\sigma(t) = M_U \cdot \varepsilon(t) - \delta M \sum_{l=1}^{n} a_l \omega_l \int_{-\infty}^{t} \varepsilon(\tau) \cdot e^{-\omega_l (t-\tau)} d\tau. \tag{16}$$

Defining a material-independent anelastic functions

$$\zeta_l(t) = \omega_l \int_{-\infty}^{t} \varepsilon(\tau) \cdot e^{-\omega_l (t-\tau)} d\tau \quad , \quad l = 1, \ldots, n \tag{17}$$

we rewrite the stress-strain relation (16) in the form

$$\sigma(t) = M_U \cdot \varepsilon(t) - \sum_{l=1}^{n} \delta M \, a_l \, \zeta_l(t). \tag{18}$$

Applying time derivative to equation (17) we get

$$\begin{aligned} \dot{\zeta}_l(t) & = \omega_l \frac{d}{dt} \int_{-\infty}^{t} \varepsilon(\tau) \cdot e^{-\omega_l (t-\tau)} d\tau \\ & = \omega_l \left[-\omega_l \int_{-\infty}^{t} \varepsilon(\tau) \cdot e^{-\omega_l (t-\tau)} d\tau + \varepsilon(t) \right] \\ & = \omega_l [-\zeta_l(t) + \varepsilon(t)] \end{aligned}$$

and

$$\dot{\zeta}_l(t) + \omega_l \zeta_l(t) = \omega_l \varepsilon(t) \quad ; \quad l = 1, \ldots, n. \tag{19}$$

Equations (18) and (19) define the time-domain stress-strain relation for the viscoelastic medium with the GMB-EK (equivalently, GZB) rheology. Its generalization to the 3D case can be found in *Kristek and Moczo* [2003]. In the velocity-stress formulation, the stress and strains are simply replaced by their time derivatives in equations (18) and (19).

5. Conclusions

[14] We have explicitly shown the equivalence of the generalized Maxwell body (GMB-EK) as defined by *Emmerich and Korn* [1987] and generalized Zener body (GZB).

[15] Two parallel streams of papers and development of incorporation of the realistic attenuation into the time-domain methods should be reviewed and compared in terms of unifying and simplifying the whole relevant theory, including curve-fitting procedures.

[16] Except for *Carcione* [2001], we have not found the correct relaxation function for the GZB. We suggest researchers that have used the formulas from the work of *Liu et al.* [1976] to check their implementations.

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